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On 't Hooft's S-matrix Ansatz for quantum black holes

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Abstract: The S-matrix Ansatz has been proposed by 't Hooft to overcome difficulties and apparent contradictions of standard quantum field theory close to the black hole horizon. In this paper we revisit and explore some of its aspects. We start by computing gravitational backreaction effects on the properties of the Hawking radiation and explain why a more powerful formalism is needed to encode them. We then use the map bulk-boundary fields to investigate the nature of exchange algebras satisfied by operators associated with ingoing and outgoing matter. We propose and comment on some analogies between the non covariant form of the S-matrix amplitude and liquid droplet physics to end up with similarities with string theory amplitudes via an electrostatic analogy. We finally recall the difficulties that one encounters when trying to incorporate non linear gravity effects in 't Hooft's S-matrix and observe how the inclusion of higher order derivatives might help in the black hole microstate counting.

Keywords: Models of Quantum Gravity, Black Holes.

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Contents

1. Introduction

The standard derivation of the Hawking radiation of black holes shows that they evaporate through (approximately) thermal radiation [1, 2]. This implies the well known problem of loss of information. In addition Hawking radiation originates in very high frequencies vacuum modes at past null infinity and gets to transplanckian energies near the horizon.

An attempt to overcome all these problems has been proposed by 't Hooft and goes under the name of S-matrix Ansatz [3]. It should also be a direct way to implement concretely the holographic principle [4, 5]. The latter seems to provide the only remedy to deal with the formation of bubbles hidden by the black hole horizon (which follows unavoidably from General Relativity). This can be seen for instance considering Schwarzschild black hole and working in the volume gauge; one discovers that larger fractions of three-volume will occupy the region beyond the horizon. It is therefore reasonable to truncate the Hilbert space of any quantum theory one has in mind to the horizon *surface* rather than to the volume of the black hole. This agrees with the fact that the black hole entropy goes like an area, not like a volume. The latter observation is normally assumed to be the starting point to justify the holographic principle.

In this paper we revisit the S-matrix Ansatz and explore some of its implications. The paper is organized as follows: in section 2 we review some aspects of the whole proposal and fix notations and conventions. We comment in particular on the definition of the horizon operators. In section 3 we examine and compute some effects of the gravitational backreaction on the properties of the Hawking radiation along the philosophy suggested by the S-matrix Ansatz. In section 4 we specify the relation between bulk and boundary fields and use it to study exchange algebras structures which have appeared before in the literature. In section 5 we propose and comment on some analogies between the non covariant form of the S-matrix amplitude and liquid droplet physics. We also point out similarities with string theory. In section 6 we recall the difficulties encountered when attempting to incorporate the non linearities of the gravitational force and observe how the inclusion of higher order derivatives might help in the black hole microstate counting. We finally summarize and give some conclusive remarks in section 6.

2. Review of the S-matrix Ansatz

In this section we revisit the S-matrix Ansatz proposal. The interested reader can find a detailed discussion and related references in the review of 't Hooft [3]. Here we simply recall and expand some points which are considered in the rest of this paper and also fix notations and conventions.

The S-matrix Ansatz basically assumes that "all physical interaction processes that begin and end with free, stable particles moving far apart in an asymptotically flat space-time, therefore *also* all those that involve the creation and subsequent evaporation of a *black hole*, can be described by one scattering matrix S relating the asymptotic outgoing states $\mid \text{out}\rangle$ to the ingoing states $\mid \text{in}\rangle$.

The starting point is to *assume* a S-matrix amplitude $\langle \text{in} \mid \text{out}\rangle$ and compute then, including the effects of the interactions δ_{in} and δ_{out}, the neighboring S-matrix elements $\langle \text{out} + \delta_{\text{out}} \mid \text{in} + \delta_{\text{in}}\rangle$.

As a first step one considers gravitational interactions only. We will see below the approximation in which the whole derivation is carried out. For the moment just recall that the effects of these interactions are basically described by gravitational shock waves: incoming particles are strongly boosted as soon as they get close to the black hole horizon and generate gravitational waves with impulsive profile, typically a Dirac delta with support on a null hypersurface. The net effect is a shift in the position of the horizon and a shift of the geodesics of the outgoing particles. Remarkably the shift can be computed explicitly [6].

More properly one considers a factorization of the S-matrix amplitude of the following form

$$S = S_{\text{out}} S_{\text{hor}} S_{\text{in}} \tag{2.1}$$

where S_{in} is supposed to relate asymptotic in-states wave packets to wave packets moving inwards very near the horizon. It only describes what goes into the black hole. S_{out} links wave packets travelling outwards very near the horizon to asymptotic out-states. It is supposed to describe all particles which come out of the black hole and to be the time reversal of S_{in}. S_{hor} represents the non trivial part of the amplitude and tells us how ingoing particles very near the horizon affect the outgoing ones.

Let us observe that this splitting should in principle give no complications[1] when considering the Rindler limit, i.e. a large mass black hole. Actually particles spend on average a time $t_A \sim 4M \ln M$ near the horizon before getting closer to it than the Planck

[1]See however [7]. We would also like to add that the location of the gluing of S_{hor} with S_{out} is relevant to establish where the virtual fluctuations due to gravitational polarization effects become Hawking particles.

length and near horizon interactions are expected to take place around this time scale. On the other hand, the average evaporation time of the black hole is $t_B \sim \mathcal{O}(M^3)$. It is reasonable to assume $t_B \gg t_A$ for large mass M and therefore one can approximate Kruskal coordinates with Rindler coordinates near the horizon. In particular one can replace the Kruskal angular coordinates $\Omega = (\theta, \phi)$ with Rindler coordinates $\tilde{x} = (x, y)$. We will use this replacement in many points in the following.

If one considers the standard Penrose diagram of Schwarzschild metric in Kruskal coordinates, 't Hooft's S-matrix puts into communication, as a consequence of backreaction effects, region I, III (i.e. the region out and in the black hole) with regions II and IV (i.e. the in and out regions of the white hole). There appears to be then a symmetry bewteen the black hole and the white hole from the very beginning. Of course this is not standard, since these two "worlds" are normally disconnected.

We come now to another *not*-trivial feature of the S-matrix Ansatz. One assumes that the ingoing and the outgoing states are fully specified once the longitudinal momenta *only* $p_{\rm in}(\tilde{x})$ and $p_{\rm out}(\tilde{x})$ are given as functions of the transverse coordinates $\tilde{x}$. In other words, $\mid \text{in}\rangle = \mid p_{\rm in}(\tilde{x})\rangle$ and $\mid \text{out}\rangle = \mid p_{\rm out}(\tilde{x})\rangle$. This approximation corresponds to an eikonal limit in which the transverse components $\Pi_{\rm in}(\tilde{x})$ and $\Pi_{\rm out}(\tilde{x})$ have been *neglected.* We discuss the nature of this limit and its relation to the standard eikonal resummation in section 6.

The relations of these momenta distributions with the components of the stress energy tensors (in Rindler coordinates) are

$$\begin{aligned}
p_{\rm in}(\tilde{x}) &= \int T_{++}(x^+, x^-, \tilde{x}) dx^+ \mid_{x^-=0} \\
p_{\rm out}(\tilde{x}) &= -\int T_{--}(x^+, x^-, \tilde{x}) dx^- \mid_{x^+=0} \\
\Pi^a_{\rm in}(\tilde{x}) &= \int T_{a+}(x^+, 0, \tilde{x}) dx^+ \\
\Pi^a_{\rm out}(\tilde{x}) &= -\int T_{a-}(x^-, 0, \tilde{x}) dx^-
\end{aligned} \tag{2.2}$$

with $a = 1, 2$ running over the transverse coordinates.

One can go then to the dual position operators via (functional) Fourier transform. Consider for instance the in-state $\mid p_{\rm in}(\tilde{x})\rangle$ (with eigenvalue $p_{\rm in}(\tilde{x})$). Its conjugate will be given by

$$\mid x_{\rm in}(\tilde{x})\rangle = \int [Dp_{\rm in}] e^{-i\int d\tilde{x} p_{\rm in}(\tilde{x}) x_{\rm in}(\tilde{x})} \mid p_{\rm in}(\tilde{x})\rangle \tag{2.3}$$

with

$$\langle x_{\rm in}(\tilde{x}) \mid p_{\rm in}(\tilde{x})\rangle = e^{i\int d\tilde{x} p_{\rm in}(\tilde{x}) x_{\rm in}(\tilde{x})}\,. \tag{2.4}$$

One assumes then standard canonical commutation relation for the in and out bases respectively, i.e.

$$[p_{\rm in}(\tilde{x}), x_{\rm in}(\tilde{x}')] = -i\delta^2(\tilde{x} - \tilde{x}') \tag{2.5}$$

$$[p_{\rm in}(\tilde{x}), p_{\rm in}(\tilde{x}')] = 0\,, \qquad [x_{\rm in}(\tilde{x}), x_{\rm in}(\tilde{x}')] = 0 \tag{2.6}$$

and analogue relations for the out sector.

Non trivial relations between in and out bases follow from taking into account gravitational backreaction effects. Explicitly one gets [3, 6]

$$\begin{aligned} x_{\rm in}(\tilde{x}) &= -\int d\tilde{x}' f(\tilde{x}-\tilde{x}') p_{\rm out}(\tilde{x}') \\ x_{\rm out}(\tilde{x}) &= \int d\tilde{x}' f(\tilde{x}-\tilde{x}') p_{\rm in}(\tilde{x}') \end{aligned} \tag{2.7}$$

where $f(\tilde{x}-\tilde{x}')$ is the shift due to gravitational shock waves which can be exactly computed. This shift goes like $\ln(\tilde{x}-\tilde{x}')$ and therefore diverges for small angular separations (where by small we mean of Planck length order). We have assumed, however, as a first approximation, to neglect the transverse components of the momentum and this means that the resolution will be always bigger than the Planck length.

What is *not* trivial is the algebra one gets once (2.7) are taken into account. Indeed one easily gets

$$[x_{\rm out}(\tilde{x}), x_{\rm in}(\tilde{x}')] = -if(\tilde{x}-\tilde{x}') \tag{2.8}$$

$$[p_{\rm in}(\tilde{x}), p_{\rm out}(\tilde{x}')] = -i\tilde{\partial}^2\delta^2(\tilde{x}-\tilde{x}')\,. \tag{2.9}$$

These relations tell us that there are non trivial correlations between ingoing and outgoing matter.

Note however the following important point: the operators $x_{\rm in(out)}(\tilde{x})$ refer to the horizon shape *not* to single particles. Thinking therefore

$$[x_{{\rm in},i}(\tilde{x}), x_{{\rm out},j}(\tilde{x}')] = -if(\tilde{x}-\tilde{x}')\delta_{ij} \tag{2.10}$$

i.e. to single particles labelled by indices i, j would be wrong. Actually all ingoing particles interact (gravitationally) with all outgoing particles. For instance, if one has two particles $1, 2$ then $x_{{\rm in},1}(\tilde{x}) - x_{{\rm out},2}(\tilde{x})$ would commute with everything and this would imply loss of information, which we want to avoid from the beginning. Several recent papers on non commutative quantum field theory pointed out a link between Moyal deformed algebras and 't Hooft's algebras. However in the former case one is referring to particle coordinates.

Therefore one has to interpret $x_{\rm in(out)}(\tilde{x})$ as

$$x_{\rm in(out)}(\tilde{x}) = \sum_i x_{{\rm in(out)},i}(\tilde{x}) \tag{2.11}$$

where the sum is over all ingoing (outgoing) particles at the *same* transverse position $\tilde{x}$.

As a consequence a single real number encodes the locations of *all* particles at a given $\tilde{x}$. All the coordinates are in this way at the right side of the horizon regardless how we shift the total $x_{in(out)}(\tilde{x})$ and given (2.11) one can reobtain all the entries in the sum in an unambiguous way [3]. From a dual perspective, the momenta distributions $p_{in(out)}(\tilde{x})$ correspond to the momenta of all particles. This is similar, as pointed out by 't Hooft and Susskind to what happens in QCD parton models. This non conventional distribution of particles and their not defined locations suggest non trivial physics, which should emerge when one zooms to planckian resolution in the transverse coordinates.

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Using then the previous relations one can show in a straightforward way that the S-matrix amplitude is given by

$$\langle p_{\rm out}(\tilde{x}) \mid p_{\rm in}(\tilde{x})\rangle = \mathcal{N} \exp\left[-i\int d^2\tilde{x}d^2\tilde{x}' p_{\rm in}(\tilde{x}')f(\tilde{x}-\tilde{x}')p_{\rm out}(\tilde{x})\right] \tag{2.12}$$

where $\mathcal{N}$ is a normalization factor which is supposed to be fixed by unitarity. Manipulating further expression (2.12) one gets

$$\begin{aligned}\langle p_{\rm out}(\tilde{x}) \mid p_{\rm in}(\tilde{x})\rangle &= \mathcal{N}\int [Dx_{\rm in}(\tilde{x})]\int [Dx_{\rm out}(\tilde{x})] \times \\ &\quad \times e^{i\int d^2\tilde{x}(-\tilde{\partial}x_{\rm out}(\tilde{x})\tilde{\partial}x_{\rm in}(\tilde{x})+p_{\rm in}(\tilde{x})x_{\rm in}(\tilde{x})-p_{\rm out}(\tilde{x})x_{\rm out}(\tilde{x}))}\,.\end{aligned} \tag{2.13}$$

Defining then $X^\mu = (x_{\rm in}, x_{\rm out})$ and $p^\mu = (p_{\rm in}, -p_{\rm out})$ one can also rewrite the amplitude in a covariant way

$$\langle {\rm in} \mid {\rm out}\rangle \sim \int d\tilde{x}\left(\frac{1}{2}(\tilde{\partial}X^\mu)^2 + p^\mu X_\mu\right). \tag{2.14}$$

Interestingly, if one imposes a static gauge on the *transverse* coordinates one gets the string theory action of Veneziano amplitudes (though with an imaginary tension despite the euclidean worldsheet). This is in turn suspected to give Nambu-Goto action once transverse components are taken into account. Even so, however, one does *not* recover a finite number of states.

The final target of the whole S-matrix Ansatz program would be thus to derive the dynamics of a *finite* number of degrees of freedom living on the horizon. The inclusion of the transverse components of the momentum is a first step in this direction. Further degrees of freedom should emerge from the inclusion of the remaining interactions, once again when one probes Planck distance scales in the transverse plane.

Thinking along similar lines 't Hooft has derived a *covariant* algebra for the horizon degrees of freedom. The building blocks are chosen to be the orientation tensors of the horizon

$$W_{\mu\nu} = \epsilon^{ij}\partial_i X_\mu \partial_j X_\nu \tag{2.15}$$

which have been shown to satisfy the covariant (local) algebra

$$[W^{\mu\alpha}(\tilde{x}), W^{\mu\beta}(\tilde{x}')] = \frac{1}{2}\delta^2(\tilde{x}-\tilde{x}')\epsilon^{\alpha\beta\mu\nu}W^{\mu\nu}(\tilde{x})\,. \tag{2.16}$$

This algebra has been derived *neglecting* second order derivatives in the embedding coordinates X^μ describing the horizon fluctuations. Despite covariant, some of its generators are not hermitian. This means at the end of the day that that one does not end up with a finite number of states. We return on these aspects in section 6.

3. Gravitational backreaction and Hawking radiation

In this section we start to examine the effects of the gravitational backreaction on the Hawking radiation. As we will see both the spectrum and the correlators display slight deviation from pure thermal radiation but these effects are transient and one needs a more powerful formalism to encode them properly. This is what the S-matrix Ansatz is indeed supposed to provide as we will discuss at the end of this section.

We consider then for simplicity a null spherically symmetric shell of matter with energy δM which falls into a Schwarzschild black hole of mass M (with $\delta M \ll M$) at some late advanced time[2] v_1. The formation time of the horizon v_0 is thus shifted to a slightly earlier time $v_0 \to v_0 + \delta v_0$ and [8]

$$\delta v_0 = -4\delta M \exp\left(\frac{v_0 - v_1}{4M}\right). \tag{3.1}$$

A light ray that originally would have reached the outside observer at some retarded time u will arrive with a time delay $\delta u(u)$

$$\delta u(u) = -4M \ln\left(1 + \frac{\delta v_0}{4M} \exp\left(\frac{u - v_0}{4M}\right)\right). \tag{3.2}$$

Notice that although the shift (3.1) is small, it can have relevant physical effects on the outgoing modes: indeed, one sees from (3.2) that the time delay $\delta u(u)$ diverges after a *finite* time $\bar{u}$

$$\bar{u} - v_0 \sim -4M \ln\left(\frac{|\ \delta v_0\ |}{4M}\right). \tag{3.3}$$

In what follows we examine the effects of the shift (3.1), i.e. a classical backreaction effect, on the properties of the outgoing Hawking radiation. Since one is not able to perform a full dynamical calculation, one repeats the same steps of Hawking derivation, this time however taking into account the shift (3.1) in the outgoing modes. As a consequence, one gets corrected Bogoliubov coefficients

$$\left\{ \begin{array}{c} \alpha^{corr}_{\omega\omega'} \\ \beta^{corr}_{\omega\omega'} \end{array} \right\} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv \exp(\mp i\omega' v) \exp(i\omega(u + \delta u)). \tag{3.4}$$

Here and in the following we use the abbreviation *corr* to remember that the coefficients which are calculated are corrected due to the presence of the shift $\delta u(u)$. One is thus computing the effect of the incoming shell of matter on the outgoing Hawking radiation.

Using the well known diffeomorphism found by Hawking [1] relating advanced and retarded coordinates

$$u(v) = v_0 - 4M \ln\left(\frac{v_0 - v}{4M}\right) \tag{3.5}$$

together with (3.2) one gets

$$\left\{ \begin{array}{c} \alpha^{corr}_{\omega\omega'} \\ \beta^{corr}_{\omega\omega'} \end{array} \right\} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0 + 4Mi\omega \ln 4M} \int_{-\infty}^{v_0} dv e^{\mp i\omega' v} (v_0 - v - D)^{-4Mi\omega} \tag{3.6}$$

where D is given by

$$D = 4\delta M \exp\left(\frac{v_0 - v_1}{4M}\right). \tag{3.7}$$

Putting now $(v_0 - v - D) = x$ one has

$$\left\{ \begin{array}{c} \alpha^{corr}_{\omega\omega'} \\ \beta^{corr}_{\omega\omega'} \end{array} \right\} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0 + 4Mi\omega \ln 4M \mp i\omega' v_0 \pm i\omega' D} \int_{-D}^{+\infty} dx e^{\pm i\omega' x} x^{-4Mi\omega}. \tag{3.8}$$

[2]As usual, $u = t - r^*$, $v = t + r^*$ and r^* is the tortoise coordinate $r^* = r + 2M \ln(r/(2M) - 1)$.

Recalling the definition of incomplete Gamma function[3] one finally obtains

$$\left\{ \begin{array}{c} \alpha^{corr}_{\omega\omega'} \\ \beta^{corr}_{\omega\omega'} \end{array} \right\} = \pm \frac{i}{2\pi\sqrt{\omega\omega'}} e^{i(\omega\mp\omega')v_0 \pm i\omega' D + 4Mi\omega \ln 4M\omega' \pm 2\pi M\omega} \Gamma(1-4Mi\omega, \pm i\omega' D) \qquad (3.10)$$

It can be easily checked that in the limit $D \to 0$ (i.e. no back-reaction) one recovers the same of [1].

In the present case, however, the Bogoliubov coefficients depend not only on the usual parameters like the black hole mass M and the time formation of the horizon v_0 but *also* on the parameters of the infalling shell, namely the energy δM and the time v_1. This suggests that the evaporation process becomes highly *dynamical* if one takes into account the backreaction, in accordance with 't Hooft's scenario.

Consider now the effects of this shift on the thermal properties of the Hawking radiation starting from the the spectrum. The Bogoliubov coefficients which give the main contribution are those with very large values of the frequency ω'. Actually if an outgoing wave reaches infinity at late times with a finite frequency ω then the incoming waves which contribute to it at early times must have a very large frequency ω' due to the large red-shift induced by the geometry of the black hole.

One is thus interested in the large ω' limit. Using the asymptotic expansion of the incomplete Gamma function[4] one gets at leading order

$$\left| \frac{\alpha^{corr}_{\omega\omega'}}{\beta^{corr}_{\omega\omega'}} \right|^2 \sim \exp(8\pi M\omega) \qquad (3.12)$$

times corrections organized in power of $1/\omega'$.

One therefore gets small deviations from the Planckian spectrum, which *also* contain explicitly the frequency ω'. These corrections, however, modify the Planckian nature of the spectrum only for a very short time.

An analogy can be made along these lines: suppose we consider some water at equilibrium in a bowl and we send in a microscopic body with high energy. The water will slightly change its temperature and thermodynamic properties and will return to equilibrium almost immediately. The quantum state of the system has changed however and one should have a powerful formalism to detect these changes.

This precisely led 't Hooft to propose the S-matrix Ansatz as a possible mechanism to encode all the changes of the black hole quantum state.

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[3]The incomplete Gamma function is defined as [9]

$$\Gamma(\alpha, z) = \int_z^\infty dt e^{-t} t^{\alpha-1} . \qquad (3.9)$$

[4]One has for $|\, z \,| \to \infty$

$$\Gamma(\alpha, z) \sim z^{\alpha-1} e^{-z} \left(1 + \frac{\alpha-1}{z} + O(z^{-2})\right) \qquad (3.11)$$

This holds properly for $|\, Arg(z) \,| < \pi/2$. In our case $z = \pm i\omega' D$ so a small real part ϵ has been added as usual to regularize producing a subleading ϵ dependent factor which can then be removed.

It is interesting to consider the stochastic properties of the outgoing radiation as well, i.e. the structure of the n-th order correlation functions. Indeed the radiation could have gaussian correlators without having a planckian spectrum or vice versa [10], something which happens for instance in quantum optics.

In the present case this means that one has to calculate not only the expectation number of the emitted particles $\langle N_\omega \rangle$ (i.e. the spectrum) but also $\langle N_\omega^2 \rangle$, $\langle N_\omega^3 \rangle$ and so on [2].

Consider for instance $\langle N_\omega^2 \rangle$. This is given by

$$\langle N_\omega^2 \rangle = \langle in \mid (b_\omega^\dagger b_\omega)(b_\omega^\dagger b_\omega) \mid in \rangle \,. \tag{3.13}$$

With the aid of the Bogoliubov transformations one gets

$$\langle N_\omega^2 \rangle = \langle N_\omega \rangle + 2(\langle N_\omega \rangle)^2 + \sum_i \alpha_{ji}\beta_{ji} \sum_k \alpha_{jk}^* \beta_{jk}^* \,. \tag{3.14}$$

Following [2] one has therefore to evaluate expressions of this form

$$\int \alpha_{\omega_1 \omega'} \beta_{\omega_2 \omega'} \,, \qquad \int \alpha_{\omega_1 \omega'}^* \beta_{\omega_2 \omega'}^* \,. \tag{3.15}$$

Using the standard Bogoliubov coefficients these integrals vanish (since proportional to $\delta(\omega_1 + \omega_2)$ and ω is positive). One has then a gaussian correlator, as expected in the case of thermal radiation. On the contrary, using the corrected Bogoliubov coefficients (3.10) these expressions do not vanish; interestingly they have the *same* order of divergence (logarithmic) as the two previous terms on the r.h.s of (3.14).

This is again in agreement with 't Hooft's scenario, implying non trivial correlations between ingoing and outgoing matter. Still, however, these correlations disappear after a very short time as just pointed out. All these transient deviations, however, suggest that the resulting Hilbert space of the system is not any more the tensor product of in and out (w.r.t the black hole) Hilbert spaces.

4. Holography at work: bulk-boundary fields and exchange algebras

The computations of the previous section suggests that the outgoing matter is thus related to the ingoing one. This fact has been used by 't Hooft to derive a non trivial algebra (2.8) satisfied by "horizon operators", whose (quite not conventional) definition has been recalled in section 2. Following the most recent developments of the holographic principle, these are *boundary* fields, namely fields which live on the boundary "screen"-the horizon-where the holographic theory is supposed to live (See for instance the recent review [11]).

In [8], however, an interesting exchange algebra satisfied by ingoing and outgoing fields was derived too. The fields which enter in this algebra, however, are now *bulk* fields and describe the higher dimensional physical world which is supposed to be encoded in the boundary description.

Consider indeed a scalar field ϕ propagating on a Schwarzschild background. One can easily show that close to the horizon the solutions of the Klein-Gordon equation split in an ingoing ϕ_{in} and an outgoing component ϕ_{out} and it was proposed in [8] that once *quantum* backreaction effects are taken into account one gets to a non trivial exchange algebra of the form

$$\phi_{\text{out}}(u,\Omega)\phi_{\text{in}}(v,\Omega') = \exp\left(-16\pi i f(\Omega,\Omega')e^{\frac{u-v}{4M}}\partial_u\partial_v\right)\phi_{\text{in}}(v,\Omega')\phi_{\text{out}}(u,\Omega) \tag{4.1}$$

where $f(\Omega,\Omega')$ is the shift due to the shock wave discussed before.

The derivation of this algebra, however, is quite not trivial and one has to make *several* assumptions in order to get to this result. One of the curious things already noticed in [8] is that the final form is symmetric in the in and out fields, while the derivation does not treat them in this way. In addition, one has to promote to operator the formation time of the horizon v_0. One assumes then a resolution in the transverse angular coordinates Ω, Ω' bigger than the Planck length (i.e. small transverse momenta); this last assumption, a sort of eikonal limit, is the same made by t'Hooft but in addition one has to take v sufficiently bigger than v_0.

In this section we would like to derive a similar result in a more direct way, using the prescription given by 't Hooft to map bulk fields into boundary ones.

The "holographic map" bulk-boundary fields proposed by 't Hooft acts in this way: take a field at some point P in the bulk and a point Q on the horizon. If the proper distance PQ is finite, then the bulk field will carry the same representations of the Hilbert space defined on the horizon. 't Hooft derives his S-matrix in the Rindler limit (which is assumed in the next considerations), so in that case all these proper distances are finite. The whole region is thus described holographically by the horizon fields. In other words, the Hilbert space associated with the degrees of freedom living on the horizon is supposed to encode the whole universe outside the black hole.

To fix ideas, consider for instance a bulk scalar field satisfying the Klein Gordon equation. In the limit of zero mass and zero transverse momenta, the bulk field $\phi(x^+, x^-, \tilde{x})$ splits into an ingoing component $\phi_{\text{in}}(x^+, \tilde{x})$ and an outgoing component $\phi_{out}(x^-, \tilde{x})$, where x^+, x^- are the usual light cone coordinates and $\tilde{x}$ represents the transverse coordinates.[5]

Using mixed Fourier transform 't Hooft has shown that up to a good approximation

$$\phi_{\text{in}}(x^+, \tilde{x}) \sim \int dk_x \exp(ik_x x^+ - ik_x x^+(\tilde{x}))\,. \tag{4.2}$$

On the r.h.s one has $x^+(\tilde{x})$, the *horizon* operator (which we denoted as $x_{\text{in}}(\tilde{x})$ in section 2). A similar, completely *symmetric* expression holds naturally for the outgoing field $x^-(\tilde{x})$ (denoted by $x_{\text{out}}(\tilde{x})$ again in section 2).

Note the non trivial relation between the bulk field ϕ_{in} and the horizon operator $x^+(\tilde{x})$. Similar complicated mappings are to be expected from the holographic principle, where a lower dimensional theory encodes higher dimensional bulk data. See for instance [12] in AdS/CFT correspondence set up. The 't Hooft eikonal limit can be rephrased also in terms of a $2+2$ splitting of Einstein gravity with two coupling constants [13]. In the case of non vanishing cosmological constant the corresponding algebras become in principle much more complicated [14].

[5]Recall that $x^+ = x + t$ and $x^- = x - t$ and $k^0 = \mid k_x \mid$ in the present situation.

We have used these expressions to obtain the commutator between ingoing and outgoing bulk fields. The computation is straightforward and one gets (using (2.1))

$$[\phi_{\rm in}(x^+,\tilde{x}),\phi_{\rm out}(x^-,\tilde{x}')] = if(\tilde{x}-\tilde{x}')\partial_+\phi_{\rm in}(x^+,\tilde{x})\partial_-\phi_{out}(x^-,\tilde{x}')\,. \qquad (4.3)$$

This non local algebra shows explicitly the complicated relation between in and out bulk fields, which would clearly commute in the case of no backreaction.

From this one has in an obvious way

$$\phi_{\rm out}(x^-,\tilde{x})\phi_{\rm in}(x^+,\tilde{x}') = (1 - if(\tilde{x}-\tilde{x}')\partial_{x^+}\partial_{x^-})\phi_{\rm in}(x^+,\tilde{x}')\phi_{\rm out}(x^-,\tilde{x})\,. \qquad (4.4)$$

Provided that, as said, the angular separation in the transverse coordinates is bigger than the Planck scale one is thus invited to exponentiate the r.h.s. obtaining thus precisely(!) the Verlinde exchange algebra (4.1) though in the Rindler limit.

The relation (4.3) seems however to us most fundamental. The exponentiation is indeed only formally justified in [8] along the lines of [15], where it was shown how to re-sum explicitly ladder diagrams in the eikonal approximation. In this situation, however, it is not completely clear to which Feynman diagrams-rules one is referring.

In addition the in-out symmetry of 't Hooft's S-matrix Ansatz is present by definition from the very beginning. As we saw in section 1 the splitting (2.1) considers the in and out sectors on equal footing. On the other hand, the symmetry in the algebra (4.1) is only recovered a *posteriori*. Actually $\phi_{\rm in}$ is supposed to evolve via a sort of Moller operator to $\phi_{\rm out}$ or $\phi_{\rm hor}$ if $v < v_0$ or $v > v_0$ respectively. Notice also that in [16] it is postulated that the energy E satisfies

$$[E, v_0] = i\,. \qquad (4.5)$$

It immediately follows (using standard Bogoliubov coefficients)

$$[E,\alpha_{\omega\omega'}] = (\omega'-\omega)\alpha_{\omega\omega'}\,, [E,\beta_{\omega\omega'}] = (\omega'+\omega)\beta_{\omega\omega'} \qquad (4.6)$$

and one can give a physical interpretation to the Bogoliubov coefficients since they carry energy E equal to $\omega'-\omega$ and $\omega'+\omega$ respectively. In the presence of backreaction, however, if one uses the corrected Bogoliubov coefficients we computed before this relation is not true any more.

5. The horizon as a membrane

As shown in section 2, by mean of Fourier transformation one can recast (2.12) in the local form (2.14). In this section, however, we are going to consider the (not trivial) expression (2.12) itself describing small fluctuations of the horizon around spherical symmetry.

We first use an analogy with liquid droplets, assimilating the black hole horizon to a droplet interface, to show how (2.12) comes out in a natural way by considering small deformations of the droplet itself. We point out then where we believe the analogy fails and list arguments which suggest to interpret the horizon as a fluctuating membrane, in agreement with standard proposals [17, 18]. We finally use an analogy with two dimensional electrostatic to interpret again (2.12) this time from a stringy perspective.

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The analogy with liquid droplets as been pointed out in a series of papers by Kastrup [19, 20]. Indeed, if one assumes- following Bekenstein old suggestion [21]-a discrete spectrum for the black hole with energy levels E_n quantized as $E_n = \sigma\sqrt{n}E_{pl}$, ($\sigma = O(1)$), and a degeneracy g^n (with g positive), the canonical partition function is clearly given by

$$Z = \sum_{n=0}^{\infty} e^{nt} e^{-\sqrt{n}x} \tag{5.1}$$

where $t = \ln g$ and $x = \beta\sigma E_{Pl}$. Kastrup has noticed that this partition function is of the same form of the one describing liquid droplets: namely the term in the exponent going like n is a bulk energy while the one going like $\sqrt{n}$ is a surface energy contribution. Using analytic continuation arguments along the lines of the elegant work of Langer [22], he also explicitly computed this partition function and found, interestingly, that its imaginary part for $g > 1$ reproduces the standard thermodynamics properties of the Schwarzschild black hole (the result holds in any dimension).

Let us then temporary use this analogy to examine the small deformations of a spherical droplet. Recall that one of the features of the droplets physics is that once the surface tension τ is given, all the properties depend on the geometry of the interface which separate the droplet from the bulk.

Suppose then that we want to consider small fluctuations of the droplet around spherical symmetry. An internal pressure p_{in} and an external pressure p_{out} act on the droplet. p_{in} will be given by $p_{\text{in}}^{eq} + \delta p_{\text{in}}(\Omega)$ where the first contribution stays for the internal pressure at equilibrium while the second contribution describes the small change in the internal pressure due to the perturbation (we allow an *angular* dependence precisely as in 't Hooft). Analogue splitting holds for the external pressure p_{out}.

At equilibrium Laplace formula must hold [23], namely

$$\tau\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = p_{\text{in}} - p_{\text{out}} \tag{5.2}$$

where R_1 and R_2 are the principal radii of curvature of the droplet. In the absence of perturbations, one obtains $p_{\text{in}}^{eq} - p_{\text{out}}^{eq} = (2\tau)/r$, where r is the radius of the droplet at equilibrium.

For small fluctuations $r \to r + \delta r(\Omega)$, at first order, one gets easily

$$(\nabla + 2)\delta r(\Omega) = \delta p_{\text{out}}(\Omega) - \delta p_{\text{in}}(\Omega) \tag{5.3}$$

where ∇ is the laplacian on the sphere. At this point it is convenient due to the symmetry of the problem to expand in spherical harmonics $\delta r(\Omega)$, $\delta p_{\text{out}}(\Omega)$ and $\delta p_{\text{in}}(\Omega)$ to get

$$\delta R^{lm}(\Omega) = \frac{\delta p_{\text{in}}^{lm}(\Omega) - \delta p_{\text{out}}^{lm}(\Omega)}{l(l+1) - 2}\,. \tag{5.4}$$

We consider now-in analogy with the 't Hooft model of the horizon- the deformation due to a force localized in a point, which is then described by a Green function $f(\Omega, \Omega')$. The latter will satisfy

$$(\nabla + 2) f(\Omega, \Omega') = -\delta^{(2)}(\Omega, \Omega') = -\sum_{l,m} Y^*_{lm}(\Omega) Y_{lm}(\Omega')\,. \tag{5.5}$$

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The sum however has to reflect in this case the physics of the droplet: the $l = 1$ component is source of divergence as can been seen from (5.4) and has to be removed. It corresponds to an infinite translation. If one assumes incompressibility too, then the $l = 0$ component has to be removed too to keep the internal volume of the droplet unchanged (we have checked however that the small angle behavior of the Green function does not change even when incompressibility is not imposed).

Assuming thus $l \geq 2$ in (5.5) one gets[6]

$$f(\theta) \sim \frac{1}{2} + \frac{2}{3}\cos(\theta) + \cos(\theta)\ln\left(\sin^2\left(\frac{\theta}{2}\right)\right). \tag{5.6}$$

Therefore the energy U (i.e. work) required to deform the droplet is given by

$$U \sim \int d\Omega d\Omega' \delta p_{\text{in}}(\Omega) f(\Omega, \Omega') \delta p_{\text{out}}(\Omega'). \tag{5.7}$$

We notice that assimilating the black hole horizon to a droplet interface one obtains basically the same expression of 't Hooft and even if the form of the shift (5.6) is different, the small angle behavior is *again* logarithmical as in 't Hooft model.

Black hole physics is however another story. We can point out the differences starting by examining 't Hooft derivation of (2.12). Before doing this, however, recall that the expression appearing in the exponent of the partition function (5.1) holds in general for large radius droplets, corresponding to compact clusters associated with low temperature physics. For smaller radius droplets, on the other hand, one gets ramified droplets and hybrid structures, which dominate the high temperature phase and *have* to be summed in the partition function [24, 25].

Despite the analogy, however, expression (2.12) derived by 't Hooft contains inputs peculiar to the black hole physics under consideration. From the shock waves analysis, indeed, one has not trivial relations among the longitudinal components of ingoing and outgoing position-momentum operators

$$\partial^2_{\tilde{x}} x_{\text{in(out)}}(\tilde{x}) = \mp p_{\text{out(in)}}(\tilde{x}). \tag{5.8}$$

Therefore our amplitude (2.12) can be also rewritten (formally) as (all fields depend on transverse coordinates $\tilde{x}$

$$\langle p_{\text{in}}(\tilde{x}) \mid p_{\text{out}}(\tilde{x}')\rangle = \int [Dx_{\text{in}}]\delta(\partial^2 x_{\text{in}} + p_{\text{in}}) e^{-i\int p_{\text{out}} x_{\text{in}}} \tag{5.9}$$

and an analogue integration can be done for the other longitudinal position operator giving

$$\langle p_{\text{in}}(\tilde{x}) \mid p_{\text{out}}(\tilde{x}')\rangle = \int [Dx_{\text{in}}][Dx_{\text{out}}] e^{-i\int \partial x_{\text{in}} \partial x_{\text{out}} + p_{\text{in}} x_{\text{out}} - p_{\text{out}} x_{\text{in}}}. \tag{5.10}$$

This is clearly a membrane like expression as recalled before. One starts from (2.12) and then Fourier transform to get (2.13). Here we just showed a different path to get to the same result. The intermediate step (5.9) shows the typical form of a path integral over a membrane where the delta takes into account all possible (in general not trivial) constraints imposed on the membrane dynamics.

[6] We used the series in 8.92 of [9].

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On general grounds, indeed, the membrane properties are different from the ones of the droplet interface. Actually an interface normally means a boundary between two phases whose fluctuations can be studied by method adapted from equilibrium critical phenomena. The statistical mechanics is normally controlled by the surface tension. Membranes, on the other hand, are composed of molecules *different* from the medium in which they are embedded, and they need not separate two distinct phases. They have in general a richer internal structure.

All of this is in agreement with 't Hooft picture, where once the transverse momenta *and* the non gravitational interactions are taken into account they are supposed to generate additional degrees of freedom living on the horizon. Despite the similarity with the energy for small deformations around spherical symmetry, it is then clear that the black hole horizon cannot be simply an interface.

Similar considerations hold in the case of the stretched horizon picture [17], though one has in this case a different set up. One is indeed interpreting the horizon as a $2+1$ dimensional membrane embedded in spacetime (more precisely a *timelike* hyper-surface). Therefore one looses the symmetry between in and out states. As a matter of fact, if one adopt the complementarity principle, from the point of view of the *static* external observer the horizon is a quantized membrane with its own degrees of freedom. One is therefore forced to use static Schwarzschild coordinates in this case. The fact that the stretched horizon has codimension one while the spacelike horizon two may have other important consequences for the counting of states, as we will see in section 6.

The expressions (2.12) and (5.5) have also another interesting analogy. They resemble the two dimensional electrostatic situation in which (2.12) can be interpreted as a scalar potential round a point charge. We want to use this analogy now to make a link with string theory amplitudes.

Suppose we start from a four point function, (below we comment on the n-point case). Imagine two ingoing and two outgoing massless particles as external lines entering an electric circular circuit. Let the coordinate along the circle be the Feynman parameter x; the typical result of a one loop integration produces

$$\int_0^1 \frac{dx}{x(1-x)} \times \text{ext. insertions}\,. \tag{5.11}$$

The contribution of the external lines is again of the form (2.12). The leading contribution from the shift is as we said logarithmic and now the "transverse coordinates" are replaced by x. Suppose we insert the four massless particles at points $0, x, 1, \infty$ (this choice will become clear in a moment). The final result is then

$$\int_0^1 \frac{dx}{x(1-x)} \exp(-2p^1p^2\ln(x) - 2p^1p^3\ln(1-x))\,. \tag{5.12}$$

Use now the Mandelstam variables s, t. One gets

$$\int_0^1 \frac{dx}{x(1-x)} \exp(-s\ln(x) - t\ln(1-x)) \tag{5.13}$$

which in turns is nothing than

$$\int_0^1 dx x^{-s-1}(1-x)^{-t-1} \tag{5.14}$$

and this is the celebrated Veneziano formula!

Recall that if one considers a n-point function for massless particles scattering in the eikonal limit (i.e. once again large s and small t Mandelstam parameters) we have been discussing, one can show [26] that the total transition amplitude nicely factorizes into the product of two particle amplitudes of the form (2.12). To fix ideas consider for instance a 6-point function. One has

$$\begin{aligned} \langle k^1, k^2, k^3 \mid p^1, p^2, p^3 \rangle = & \int dx^1 dx^2 dx^3 \langle k^{(1)}, k^{(2)} \mid x^{(1)}, x^{(2)} \rangle \times \\ & \times \langle x^{(2)}, k^{(3)} \mid p^{(2)}, x^{(3)} \rangle \langle x^{(1)}, x^{(3)} \mid p^{(1)}, p^{(3)} \rangle \end{aligned} \tag{5.15}$$

where k, p stay for incoming and outgoing momenta respectively of the particles $1, 2, 3$. We are currently trying to understand what these generalizations correspond from stringy perspective to see if more information can be extracted [35].

6. On the inclusion of transverse momenta

As recalled in section 1, the horizon operator algebra once written in a covariant way should give the correct counting of the black hole microstates. Promoting the orientation tensors (2.15) of the horizon (see additional comments below in this section) to operators, 't Hooft has obtained a covariant algebra (i.e. Lorentz invariant) and the generators of these algebra are supposed to correspond to different cell-domains defined on the horizon. Remarkably each cell carries quantum angular momentum numbers. Unfortunately some of the generators are not hermitian, so one does not end up with a finite number of states to satisfy the entropy bound.

The main difficulty to overcome is thus how to include in a consistent way in the S-matrix Ansatz proposal the transverse components of the gravitational interactions to get covariance and a finite number of states at the same time.

What is not trivial in particular is the fact that that the transverse momentum is not an independent variable. Indeed, if $Q_{\rm in}(\tilde{x})$ is any operator valued function of the transverse coordinates $\tilde{x}$, one has for the incoming transverse momentum $\Pi^a_{\rm in}(\tilde{x})$(here $\partial_{tr} = \tilde{\partial}$)

$$[\Pi^a_{\rm in}(\tilde{x}), Q_{\rm in}(\tilde{y})] = -i\delta^{(2)}(\tilde{x}-\tilde{y})\partial^a_{tr} Q_{\rm in}(\tilde{x})\,. \tag{6.1}$$

One can show that an operator satisfying such properties is given by composing ingoing longitudinal position and momentum operators as follows

$$\Pi^a_{\rm in}(\tilde{x}) = p_{\rm in}(\tilde{x})\partial^a_{tr} x_{\rm in}(\tilde{x})\,. \tag{6.2}$$

The same holds of course for the outcoming transverse momenta.

In analogy with the longitudinal components of the momentum one would expect

$$[\Pi^a_{\rm in}(\tilde{x}), \Pi^b_{\rm out}(\tilde{y})] = -i\delta^{ab}\partial^2_{tr}\delta^{(2)}(\tilde{x}-\tilde{y})\,. \tag{6.3}$$

On the other hand one gets a much more complicated expression. We have found explicitly

$$[\Pi^a_{\rm in}(\tilde{x}), \Pi^b_{\rm out}(\tilde{y})] = ip_{\rm in}(\tilde{x})p_{\rm out}(\tilde{y})\partial^a_{tr}\partial^b_{tr}f(\tilde{x}-\tilde{y}) - i\partial^2_{tr}\delta(\tilde{x}-\tilde{y})\partial^b_{tr}x_{\rm out}(\tilde{y})\partial^a_{tr}x_{\rm in}(\tilde{x})\,. \tag{6.4}$$

This algebra is again non local and in addition it contains higher order derivatives. In particular, transverse derivatives of the Green function $f(\tilde{x}-\tilde{x}')$ suggest the presence of leading order correction terms to the eikonal approximation.

These corrections have been analyzed in detail [27] and it has been shown that they are related to gravitational emission (one loop quantum corrections and two loops classical corrections), not really to an external metric like the leading order shift $f(\tilde{x},\tilde{x}')$. They represent a sort of “refraction” effect in the transverse plane and measure how much the angle that a geodesic forms with $x^+ = 0$ for instance changes when the hypersurface $x^+ = 0$ is crossed. What is not trivial is the fact that there might be points in the transverse place with suffer a discontinuity produced by the shock *but* no refraction and/or viceversa.

Therefore, when including second order derivatives of the X^μ embedding functions of our horizon-membrane (which have been always neglected up to now), it is reasonable to expect then that the relations (2.8) do not hold any more even before including the transverse momenta. Unfortunately it appears not obvious at all how to modify them.

It is interesting however to figure out, at least from a qualitative point of view, what happens when one considers a more general situation. Up to this moment, as said, second order derivatives of the embedding functions X^μ have been neglected. There are however important differences which could be relevant for the microstate counting once included.

First we want however to point out some differences between 't Hooft and standard “membrane paradigm” approaches.

To start 't Hooft is considering the embedding of the two dimensional surface, the spacelike horizon, into four dimensional target spacetime. The spacelike horizon is the intersection point of past and future horizons and it is expected to be a very complicated curved sub-manifold because of back-reaction effects. As we have seen one chooses a static gauge for the transverse coordinates x^1, x^2. Normally, however, the static gauge is imposed on x^0, x^3. In principle a double Wick rotation should bring us to the standard situation, as suggested by 't Hooft. In the latter case one imposes the static gauge[7] on the world-volume coordinates and the transverse ones are expected to be the analogue of the $x^\pm$ we have been discussing once the rotations have been performed. This is similar to what happens in recent D-brane models, where the non commutativity is in the transverse directions. We imagine therefore to have performed such Wick rotations and be in the standard case.

Secondly, in the usual membrane paradigm and its “quantum evolutum” stretched horizon one has in mind a $2+1$ membrane embedded in $3+1$ spacetime, while here the spacelike horizon is *two* dimensional. This is due to the definition (2.2) of incoming and outgoing momenta where one *restricts* to $x^\pm = 0$ respectively. Instead of codimension one

[7]I thank Shmuel Elitzur for discussions on these topics.

we have codimension two and this will be relevant when considering actions with second order derivatives of the embedding functions X^μ.

In general, if one requires reparametrization invariance such an action[8] can only depend of the induced metric on the worldvolume $h_{ij} = g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu$ and the extrinsic curvature $K_{ij} = n_{\mu;\nu}\partial_i X^\mu \partial_j X^\nu$. Therefore one has an expansion of operators constructed out of these two quantities and their derivatives. Similar models have indeed already been considered also in the case of black holes physics [28].(For a recent though slightly different proposal [29]).

Having in mind the S-matrix Ansatz, we now simply want to analyze the momenta conijugate to the embedding functions $X^\mu(\xi)$.

We start with the Nambu Goto action, which by dimensional analysis turns out to be the leading order term in a candidate effective action and is supposed to emerge from the S-matrix Ansatz when neglecting indeed second order derivatives.

Before doing this notice that if the codimension $\tilde{d}$ of the worldvolume with respect to the bulk (i.e. the target space in which the worldvolume is embedded) is greater than one, there will be $\tilde{d}$ spacelike vector field n^r , $(r = 1, \dots, \tilde{d})$, normal to the worldvolume. As a consequence there will be $\tilde{d}$ extrinsic curvature tensors K^r_{ij}, one along each normal. This will clearly affect the counting of the horizon degrees of freedom, since more orientations are now possible. It also in general fits with the holographic principle, where the bulk-boundary codimension can change the holographic map rule.

As an example consider the model quadratic in the mean extrinsic curvature [30, 31] with action

$$S = \int_W d^{d+1}\xi \sqrt{-h} K^2 \,. \tag{6.5}$$

It has been used in the context of rigid strings and its action can also be rewritten as

$$\int \sqrt{-h} K^{i,r}_j K^{j,r}_i = \int \sqrt{-h} h^{ij} \partial_i W_{\mu\nu} \partial_j W_{\mu\nu} = \int \sqrt{-h} h^{ij} \nabla_i n_r \nabla_j n_r \tag{6.6}$$

with $\partial_i \partial_j X^\mu = \Gamma^k_{ij} \partial_k X^\mu + K^r_{ij} n^\mu_r$. Notice the appearance of the orientation tensor $W^{\mu\nu}$ which has been indeed proposed by 't Hooft as a candidate to describe the horizon algebra. But in 't Hooft's picture the codimension is two, so these objects should carry an additional index corresponding to two different normals. This might in principle give a more refined microstate counting. It appears however including second order derivatives of the embedding functions as showed. These contributions are indeed neglected in 't Hooft's derivation and we suspect that this is one of the reasons of the unbounded number of states given in the resulting algebra.

We *insist* on this point since the horizon orientation is important with respect to the *observer* we are considering. A concrete example is given in the $2 + 1$ dimensional case,

[8] We adopt the following conventions and notations: the target spacetime is d+2 dimensional with metric $g_{\mu\nu} = \eta_{\mu\nu}$ with $\mu, \nu = 1, \dots d + 2$. h_{ij} is the metric of the $d + 1$ dimensional worldvolume W, with $i, j = 1, \dots, d + 1$. We split then the worldvolume coordinates ξ^i as follows $\xi^i = (\tau, \sigma^a)$, i.e. time-spatial coordinates, with $a, b = 1, \dots d$. We denote with Σ the spacelike surface spanned by the σ^a coordinates and with q_{ab} the induced metric on Σ.

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where to make the algebra covariant one performs at some point a coordinate transformation with a jacobian whose signs tells us if one is inside or outside the horizon [32].(See also [33])

Let us finally include second order derivatives of the embedding functions. For simplicity we choose target flat spacetime which fits nicely also with the splitting (2.1) of the S-matrix for the near horizon region we are interested.

We consider first the Nambu-Goto action. One has

$$S_{NG} = T \int_W d^{d+1}\xi \sqrt{-h} \tag{6.7}$$

where $h_{ij} = \eta_{\mu\nu}\partial_i X^\mu \partial_j X^\nu$ is the induced metric on the worldvolume and $X^\mu(\xi^i) = X^\mu(\tau, \sigma^a)$ are the embedding functions with the worldvolume splitting of coordinates (τ, σ^a).

Varying the action w.r.t. the embedding functions X^μ one gets

$$\delta S_{NG} = T \int_W d^{d+1}\xi \nabla_j \left(\sqrt{-h} h^{ij} \partial_i X^\mu \delta X_\mu\right) - T \int_W d^{d+1}\xi \nabla_j \left(\sqrt{-h} h^{ij} \partial_i X^\mu\right) \delta X_\mu \tag{6.8}$$

where the first term on the r.h.s is a total derivative. One is then invited to apply Stokes theorem[9] to get

$$\delta S_{NG} = -T \int_\Sigma d^d\sigma \sqrt{-q} \lambda_j h^{ij} \partial_i X^\mu \delta X_\mu - T \int_W d^{d+1}\xi \nabla_j \left(\sqrt{-h} h^{ij} \partial_i X^\mu\right) \delta X_\mu \tag{6.10}$$

On the other hand one has for a generic action depending only on first order derivatives of X^μ

$$\delta S = \int_W d^{d+1}\xi \sqrt{-h} \frac{\delta S}{\delta X_\mu} \delta X^\mu + \int_\Sigma d^d\sigma p_\mu \delta X^\mu \tag{6.11}$$

where p^μ is the conjugated momentum to X^μ.

Comparing (6.10) with (6.11), it follows

$$p^\mu = -T\sqrt{-q} \lambda^i \partial_i X^\mu \,. \tag{6.12}$$

One sees that in this case the momentum has *only* tangential components to the worldvolume.

We now consider the case in which second order derivatives of the embedding functions X^μ are present. Recall from standard classical mechanics that if we have an action like [34]

$$S = \int_M dt L(q, \dot{q}, \ddot{q}) \tag{6.13}$$

[9]Recall that for a vector B^i Stokes theorems goes as follows

$$\int_W d^{d+1}\xi \nabla_i (\sqrt{-h} B^i) = - \int_\Sigma d^d\sigma \sqrt{-q} \lambda_i B^i \tag{6.9}$$

where λ_i is a timelike vector normal to Σ and $q_{ab} = h_{ij}\partial_a X^i \partial_b X^j$ with $\partial_a = \partial/\partial\sigma^a$ is the induced metric on Σ

one will have not only a momentum p conjugated to q but also a momentum Π conjugated to $\dot{q}$. They are easily found to be

$$\Pi = \frac{\delta L}{\delta \ddot{q}}\,, \qquad p = \frac{\delta L}{\delta \dot{q}} - \partial_t \left(\frac{\delta L}{\delta \ddot{q}}\right). \tag{6.14}$$

Consider therefore the most generic worldvolume action containing second order derivatives of the embedding functions. As remarked before to assure reparametrization invariance it has to be of the form

$$S = \int_W d^{d+1}\xi \sqrt{-h} L(h_{ij}, K_{ij})\,. \tag{6.15}$$

We have assumed for simplicity codimension one, otherwise, as pointed out before, the extrinsic curvature will carry additional indices. The general variation is given by

$$\delta S = \int_W d^{d+1}\xi \ \sqrt{-h}\frac{\delta L}{\delta X^\mu}\delta X_\mu + \int_\Sigma \Pi^\mu \delta \dot{X}_\mu + p^\mu \delta X_\mu \tag{6.16}$$

where one reads from the boundary term the momenta p^μ and Π^μ. On the other hand, from the action (6.15) one gets (integrating by parts and using $\delta K_{ij} = -n^\mu \nabla_i \nabla_j \delta X_\mu$)

$$\delta S = \int_W d^{d+1}\xi \sqrt{-h} \nabla_j S^j + \text{bulk} - e.o.m \tag{6.17}$$

with

$$S^j = \left(L h^{ij}\partial_i X^\mu - 2\frac{\delta L}{\delta h_{ij}}\partial_i X^\mu + \nabla_i\left(\frac{\delta L}{\delta K_{ij}} n^\mu\right)\right)\delta X_\mu - \frac{\delta L}{\delta K_{ji}} n^\mu \nabla_i \delta X_\mu \tag{6.18}$$

where n^μ is a spacelike unit normal to the worldvolume. Once again we can then apply Stokes theorem and find explicit expressions for π^μ and Π^μ (making manifest the dependence on $\dot{X}^\mu$). We stop here however to observe that that the momentum p^μ, contrary to the Nambu-Goto situation, has now *also* components which are *not* tangential to the worldvolume.

When deriving the algebra (2.16) these components, which as shown come about when including second order derivatives, have not been considered and this might be one of the reasons for which one does not get a finite number of states [35]. There are also the Π^μ components to be taken into account and we do not have a clear interpretation of their role at the moment.

7. Concluding remarks

In this paper we have revisited 't Hooft S-matrix Ansatz for quantum black holes. We have analyzed the whole proposal from different perspectives which we now summarize.

We started in section 3 by considering the effects of the gravitational backreaction on the Hawking radiation. We have seen that that there is shift of the phase of the outgoing wavepackets which has physical effects. Of course if outgoing particles were eigenstates with respect to Kruskal momentum, this effect would be irrelevant. However this is *not* the case in general and therefore the quantum state *changes*.

If one then asks questions concerning the nature of the Hawking radiation, these effects do not change anything at long times (except very short and small fluctuations during the evaporation of the black hole which we have computed for a spherically symmetric shell): therefore one still has thermal spectrum and gaussian correlators, i.e. total absence of correlations among wave-packets. Indeed Hawking radiation simply originates now in a different region that it would have if particles had not been send into the black hole.

The S-matrix Ansatz suggests that instead of a density matrix, which always gives thermal spectrum when tracing over an inner region of space-time, one should try on the contrary to construct such a detailed and precise theory able to *produce* this S-matrix itself. Remember that the S-Matrix is *assumed* and one looks at the back-reaction effects on it. This is of course not an easy task; quoting [17] "...if the microscopic laws were known, computing an S matrix would , according to this view, be as daunting as computing the scattering of laser light from a chunk of black coal".

In section 4 we have considered the map bulk boundary fields and applied it to have a clear derivation non commutative algebras between ingoing and outgoing bulk fields just using 't Hooft's results. These non-commutative structures seem to be particularly relevant in the context of black hole physics since they represent first of all a natural manifestation of the complementarity principle [3, 17] and secondly should help in reducing the degeneracy of states of the near horizon region.

In section 5 we have made some analogies with liquid droplets and two dimensional electrostatic. We have seen that despite some analogies with droplet physics and the intriguing fact that for high temperature non compact clusters have to be included (implying more coarse graining and therefore more information storing) the horizon fluctuations according to the S-matrix necessitate a membrane-like picture. We are still exploring the analogy with two dimensional electromagnetism to see if one can make more contact with stringy aspects. In particular we are trying to understand what kind of "electric circuit" should be used for contributions beyond the eikonal.

In section 6 we have pointed out some of the difficulties concerning the inclusion of the transverse momenta. It is at this level that all the non-linearities of the gravity are expected to play a role. We have seen that the inclusion of higher derivatives could be in principle a good tool to guess the right horizon algebra. The inclusion of transverse momenta is supposed to produce then a granular structure on the horizon. This should come out from the horizon algebra itself even if not in an easy way of course.

We are therefore first trying to understand the connection between the role of the transverse degrees of freedom and the interesting result that the entropy rate production of a black hole behaves like the one of a *one* dimensional system [36], displaying dimensional reduction in analogy with the entropy area law for black holes.[10]

[10]We thank Jacob Bekenstein for introducing us to the ideas expressed in [36] and Gerard 't Hooft for the suggestion just mentioned.

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References

[1] S.W. Hawking, *Particle creation by black holes*, *Commun. Math. Phys.* **43** (1975) 199.

[2] S.W. Hawking, *Breakdown of predictability in gravitational collapse*, *Phys. Rev.* **D 14** (1976) 2460.

[3] G. 't Hooft, *The scattering matrix approach for the quantum black hole: an overview*, *Int. J. Mod. Phys.* **A 11** (1996) 4623 [gr-qc/9607022] and references therein.

[4] G. 't Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026.

[5] L. Susskind, *The world as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089].

[6] T. Dray and G. 't Hooft, *The gravitational shock wave of a massless particle*, *Nucl. Phys.* **B 253** (1985) 173.

[7] N. Itzhaki, *Some remarks on't Hooft's S-matrix for black holes*, hep-th/9603067.

[8] Y. Kiem, H. Verlinde and E. Verlinde, *Black hole horizons and complementarity*, *Phys. Rev.* **D 52** (1995) 7053 [hep-th/9502074].

[9] I.S. Gradshteyn and I.M. Ryzhik, *Tables of integrals series and products*, Academon Press 1965.

[10] D.W. Sciama, P. Candelas and D. Deutsch , *Quantum field theory, horizons and thermodynamics*, *Adv. in Phys.* **30** 3 (1981) 327.

[11] J. de Boer, L. Maoz and A. Naqvi, *Some aspects of the AdS/CFT correspondence*, hep-th/0407212.

[12] T. Banks, M.R. Douglas, G.T. Horowitz and E.J. Martinec, *AdS dynamics from conformal field theory*, hep-th/9808016.

[13] H. Verlinde and E. Verlinde, *Scattering at planckian energies*, *Nucl. Phys.* **B 371** (1992) 246 [hep-th/9110017].

[14] G. Arcioni, S. de Haro and M. O'Loughlin, *Boundary description of planckian scattering in curved spacetimes*, *J. High Energy Phys.* **07** (2001) 035 [hep-th/0104039].

[15] H.D.I. Abarbanel and C. Itzykson, *Relativistic eikonal expansion*, *Phys. Rev. Lett.* **23** (1969) 53.

[16] K. Schoutens, H. Verlinde and E. Verlinde, *Quantum black hole evaporation*, *Phys. Rev.* **D 48** (1993) 2670 [hep-th/9304128].

[17] L. Susskind, L. Thorlacius and J. Uglum, *The stretched horizon and black hole complementarity*, *Phys. Rev.* **D 48** (1993) 3743 [hep-th/9306069].

[18] R.H. Price and K.S. Thorne, *Membrane viewpoint on black holes: properties and evolution of the stretched horizon*, *Phys. Rev.* **D 33** (1986) 915.

[19] H.A. Kastrup, *Canonical quantum statistics of Schwarzschild black holes and ising droplet nucleation*, *Phys. Lett.* **B 419** (1998) 40 [gr-qc/9710032].

[20] H.A. Kastrup, *Schwarzschild black hole quantum statistics from Z(2) orientation degrees of freedom and its relations to ising droplet nucleation*, *Annalen Phys.* **9** (2000) 503–522 [gr-qc/9906104].

[21] J.D. Bekenstein, *The quantum mass spectrum of the Kerr black hole*, *Lett. Nuovo Cim.* **11** (1974) 467.

[22] J.S. Langer, *Theory of the condensation point*, *Ann. Phys. (NY)* **41** (1967) 108.

[23] L. Landau and E. Lifschitz, *Fluid Mechanics*, Pergamon, New York 1987.

[24] C. Domb, *Metastability and spinodals in the lattice gas model*, *J. Phys. A: Math. Gen.* **9** (1976).

[25] K. Binder, Theory of first-order phase transitions, *Rep. Prog. Phys.* **50** (1987) 783.

[26] G. 't Hooft, *On the factorization of universal poles in a theory of gravitating point particles*, *Nucl. Phys.* **B 304** (1988) 867.

[27] D. Amati, M. Ciafaloni and G. Veneziano, *Planckian scattering beyond the semiclassical approximation*, *Phys. Lett.* **B 289** (1992) 87 and references therein.

[28] A. Buonanno, M. Gattobigio, M. Maggiore, L. Pilo and C. Ungarelli, *Effective lagrangian for quantum black holes*, *Nucl. Phys.* **B 451** (1995) 677 [gr-qc/9504020].

[29] N. Itzhaki, *The horizon order parameter*, hep-th/0403153.

[30] A.M. Polyakov, *Fine structure of strings*, *Nucl. Phys.* **B 268** (1986) 406.

[31] H. Kleinert, *The membrane properties of condensing strings*, *Phys. Lett.* **B 174** (1986) 335.

[32] G. 't Hooft, *Transplanckian particles and the quantization of time*, *Class. and Quant. Grav.* **16** (1999) 395 [gr-qc/9805079].

[33] S. de Haro, *Planckian scattering and black holes*, *J. High Energy Phys.* **10** (1998) 023 [gr-qc/9806028].

[34] See for instance E.T. Whittaler, *Analytical dynamics of particles and rigid bodies*, Cambridge University Press, Cambridge 1964.

[35] G. Arcioni, work in progress.

[36] J.D. Bekenstein and A.E. Mayo, *Black holes are one-dimensional*, *Gen. Rel. Grav.* **33** (2001) 2095 [gr-qc/0105055].

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